[11:00-11:20] Frequency response for discrete-time systems

If a complex sinusoidal signal is input to an LTI system, the output is another complex sinusoid with the same frequency; no new frequencies are created. The sinusoid that is output has an amplitude scaled by the magnitude response $\left|H\left(e^{j\hat{\omega}}\right)\right|$ and has its phase shifted by the phase response $\angle H\left(e^{j\omega}\right)$.

$$\begin{matrix} x[n] \\ X(e^{j\hat{\omega}}) \end{matrix} \rightarrow \begin{matrix} y[n] = h[n] * x[n] \\ Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})X(e^{j\hat{\omega}}) \end{matrix}$$

$$H\left(e^{j\hat{\omega}}\right) = \left|H\left(e^{j\hat{\omega}}\right)\right| e^{j\angle H\left(e^{j\omega}\right)}$$

We can design the LTI system to apply gain—amplification, no change, attenuation, or zerod out—to different frequencies.

$$\left|H\left(e^{j\hat{\omega}}\right)\right| = \begin{cases} > 1 & \text{amplification} \\ 1 & \text{no change} \\ (0,1) & \text{attenuation} \\ 0 & \text{zero out} \end{cases}$$

$$\underbrace{\text{group delay}(\hat{\omega})}_{\text{units of samples}} = \frac{-d}{d\hat{\omega}} \angle H\left(e^{j\hat{\omega}}\right)$$

For many applications (e.g audio), we want group delay to be constant as a function of frequency; otherwise, components at different frequencies (e.g. different instruments) will lose synchronization.

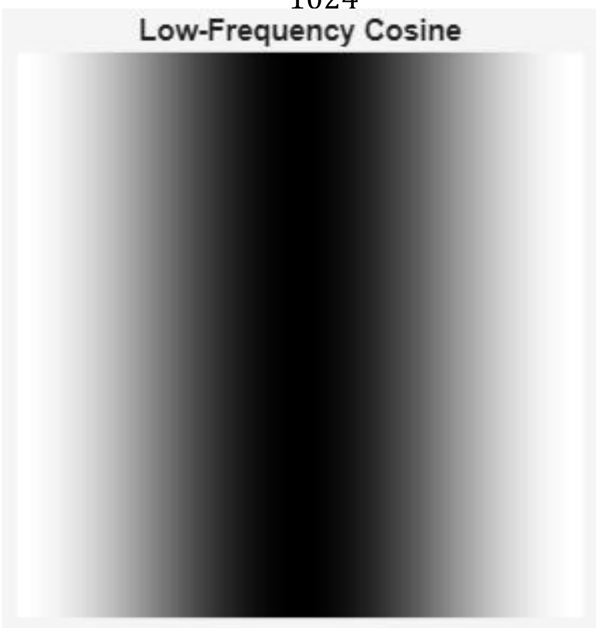
[11:20-11:30] Frequency response of common systems

	$\left\|H\left(e^{j\hat{\omega}}\right)\right\|$	$\angle H\left(e^{j\hat{\omega}}\right)$
Ideal delay $y[n] = x[n-1]$	1	$-\omega$
Two sample average $y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$	$\cos\left(\frac{\hat{\omega}}{2}\right)$	$-\frac{\hat{\omega}}{2}$
First order difference $y[n] = \frac{1}{2}x[n] - \frac{1}{2}x[n-1]$	$\begin{cases} \sin\left(\frac{\hat{\omega}}{2}\right) & 0 \le \hat{\omega} < \pi \\ -\sin\left(\frac{\hat{\omega}}{2}\right) & -\pi \le \hat{\omega} < 0 \end{cases}$	$\begin{cases} \frac{\pi}{2} - \frac{\hat{\omega}}{2} & 0 \le \hat{\omega} < \pi \\ -\frac{\pi}{2} - \frac{\hat{\omega}}{2} & -\pi \le \hat{\omega} < 0 \end{cases}$

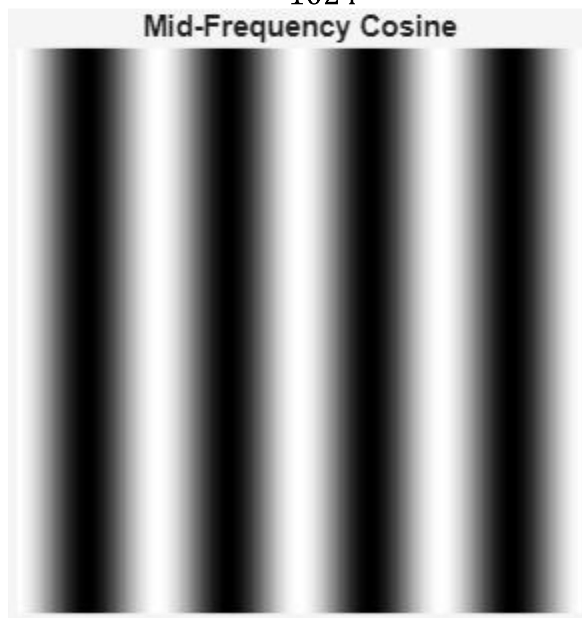
[11:30-12:00] Image processing demo

If a discrete-time cosine completes one full cycle over the course of 1024 samples, the discrete-time frequency is $2\pi \frac{1}{1024} = \frac{\pi}{512}$ radians per sample. If four cycles are completed, the discrete-time frequency would be $2\pi \frac{4}{1024} = \frac{\pi}{128}$, and so on.

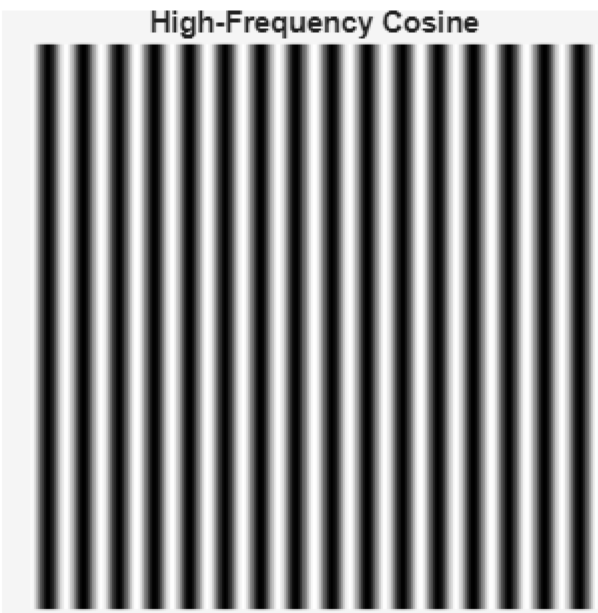
$$2\pi \frac{1}{1024}$$



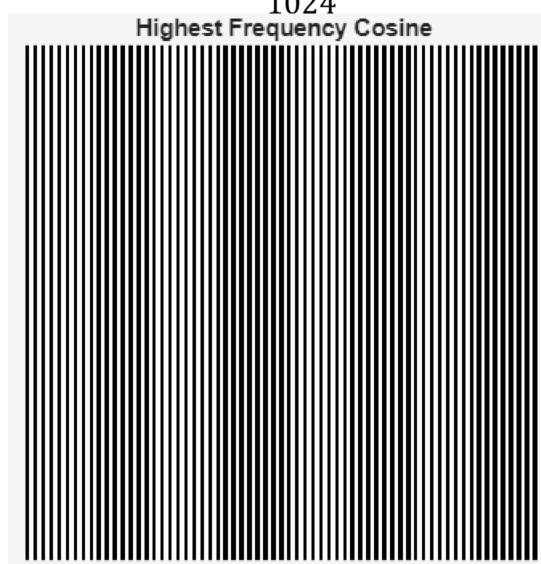
$$2\pi \frac{4}{1024}$$



$$2\pi \frac{16}{1024}$$



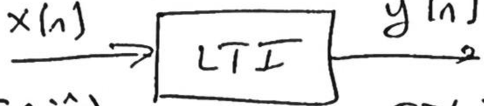
$$2\pi \frac{512}{1024}$$



(aliasing occurs when rendered at this resolution)

Review 10/28/25

$x[n]$



$y[n] = h[n] * x[n]$

$X(e^{j\hat{\omega}})$

$h[n]$
impulse response

$Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})\, X(e^{j\hat{\omega}})$

we can design the LTI system to apply gain — amplification, no change, attenuated, zeroed out — to different frequencies

$$H(e^{j\hat{\omega}}) = |H(e^{j\hat{\omega}})|\, e^{j\angle H(e^{j\hat{\omega}})}$$

$$|H(e^{j\hat{\omega}})| = \begin{cases} >1 & \text{amplification} \\ 1 & \text{no change} \\ (0,1) & \text{attenuation} \\ 0 & \text{zero out} \end{cases}$$

$$\text{Group Delay}(\hat{\omega}) = -\frac{d}{d\hat{\omega}} \angle H(e^{j\hat{\omega}})$$

in units of samples

imageRangesCosines.m demo

$y_1[n] = \cos(\hat{\omega}_1 n)$ for the first cosine

does not include a scaling by 127.5 and offset of 127.5

$\hat{\omega}_1 = \frac{2\pi}{1024}$

$y_2[n] = \cos(\hat{\omega}_2 n)$ second cosine

$\hat{\omega}_2 = \frac{2\pi}{1024}(4)$

$\hat{\omega}_3 = \frac{2\pi}{1024}(16)$ third cosine

$\frac{16}{1024} = \frac{1}{64}$ ← # samples in one cosine period

$\hat{\omega}_4 = \frac{2\pi}{1024}(512) = \pi$ ← highest freq. in DT

Aside: $\cos(\pi n) = (-1)^n$

$\sin(\pi n) = 0$